_1	/13
2	/18
3	/19
4	/18
5	/17
6	/15

MASSACHUSETTS INSTITUTE OF TECHNOLOGY DEPARTMENT OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

6.191 Computation StructuresSpring 2023

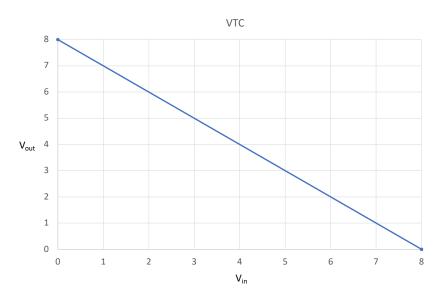
Quiz #1

Name		Athena logi	in name	Score
Solutions				
Recitation section				
☐ WF 10, 34-302 (Alexandra)	□ WF 2, 34-302 ((Boom)	☐ opt-ou	ıt
☐ WF 11, 34-302 (Alexandra)	□ WF 3, 34-302 ((Boom)		
☐ WF 12, 34-302(Georgia)	□ WF 12, 35-308	(Keshav)		
☐ WF 1, 34-302 (Georgia)	□ WF 1, 35-308	(Keshav)		
` `				

Please enter your name, Athena login name, and recitation section above. Enter your answers in the spaces provided below. Show your work for potential partial credit. You can use the extra white space and the back of each page for scratch work.

Problem 1. Digital Abstraction (13 points)

(A) (3 points) The Voltage Transfer Curve of Device X is shown below.



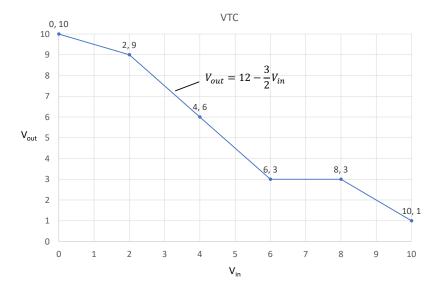
Give a valid specification for Device X or explain why no such specification exists.

$$V_{OL}$$
: ____; V_{IL} : ____; V_{IH} : ____; V_{OH} : ____

Or explain why a valid specification for Device X does not exist:

No such specification exists. The voltage transfer curve shown has no gain. There is no possible specification for Device Y with positive noise immunity.

Device Y has the Voltage Transfer Curve shown below. Device Y is an inverter.



(B) (6 points) For each partial specification below, complete the table by filling in the missing entries that **maximize the noise immunity given the existing constraints**. If the voltages in a given row cannot be part of valid specification, fill the entries for that row with "X"s. Remember, a valid specification must have positive noise immunity.

	V_{OL}	$\mathbf{V}_{\mathbf{IL}}$	V _{IH}	Voh	Noise Immunity
Spec 1	1	X	X	10	X
Spec 2	3	X	X	9	X
Spec 3	3	3.6	6	6.6	0.6
Spec 4	3	X	X	6	X

(C) (4 points) Explain why each of these specifications is valid or invalid.

For spec 1, V_{IL} must be 0 to guarantee V_{IH} of 10, but then V_{OL} would be $> V_{IL}$.

For spec 2, V_{IL} must be at most 2 to guarantee V_{IH} of 9, but then V_{OL} would be $> V_{IL}$.

For spec 3, the given specification follows that $V_{OL} < V_{IL} < V_{IH} < V_{OH}$. A V_{IL} of 3.6 will guarantee a high output of V_{OH} , or 6.6. A V_{IH} of 6 will guarantee a low output of V_{OL} , or 3. There is a positive noise immunity of 0.6.

For spec 4, V_{IL} must be 4 to guarantee V_{OH} of 6. V_{IH} must be 6 to guarantee a V_{OL} of 3. This leads to zero noise immunity, so it is not a valid specification.

Problem 2. Boolean Algebra (18 points)

- (A) (8 points) Zoomba, a diligent student in 6.191, is trying to simplify some combinational logic. Help Zoomba by **finding a minimal sum-of-products** expression for each of the following Boolean expressions. (Note: These expressions can be reduced into a minimal SOP by repeatedly applying the Boolean algebra properties we saw in lecture.) Make sure your answer is in minimal-sum-of products form.
 - $1. \quad \overline{xy(x+\bar{y})+\bar{y}}$

$$\overline{xy(x+\bar{y})+\bar{y}} = \overline{xyx+xy\bar{y}+\bar{y}} = \overline{xy+\bar{y}} = \overline{x+\bar{y}} = \bar{x}y$$

2. $(a+b)(a+\bar{b})+\bar{a}\bar{b}$

$$(a+b)(a+\bar{b}) + \bar{a}\bar{b} = aa + ab + ba + b\bar{b} + \bar{a}\bar{b} = a + ab + \bar{a}\bar{b}$$
$$= a(1+b) + \bar{a}\bar{b} = a + \bar{a}\bar{b} = a + \bar{b}$$

3. $\bar{x}\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}yz + xyz + xy\bar{z}$

$$\bar{x}\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}yz + xyz + xy\bar{z} = \bar{x}\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}y\bar{z} + \bar{x}yz + xyz + xy\bar{z}$$

$$= \bar{x}\bar{z}(y + \bar{y}) + \bar{x}y(z + \bar{z}) + xy(z + \bar{z})$$

$$= \bar{x}\bar{z} + \bar{x}y + xy$$

$$= \bar{x}\bar{z} + y(x + \bar{x})$$

$$= \bar{x}\bar{z} + y$$

4. $(ab + \bar{a}\bar{b})c + b\bar{c} + \bar{a}\bar{c}$

$$(ab + \bar{a}\bar{b})c + b\bar{c} + \bar{a}\bar{c} = abc + \bar{a}\bar{b}c + b\bar{c} + \bar{a}\bar{c}$$

$$= abc + \bar{a}\bar{b}c + (ab\bar{c} + \bar{a}b\bar{c}) + (\bar{a}b\bar{c} + \bar{a}\bar{b}\bar{c})$$

$$= abc + \bar{a}\bar{b}c + ab\bar{c} + \bar{a}b\bar{c} + \bar{a}\bar{b}\bar{c}$$

$$= ab(c + \bar{c}) + \bar{a}\bar{c}(b + \bar{b}) + \bar{a}\bar{b}(c + \bar{c})$$

$$= ab + \bar{a}\bar{c} + \bar{a}\bar{b}$$

Alternate answer:

$$(ab + \bar{a}\bar{b})c + b\bar{c} + \bar{a}\bar{c} = abc + \bar{a}\bar{b}c + b\bar{c} + \bar{a}\bar{c}$$

$$= abc + \bar{a}\bar{b}c + (ab\bar{c} + \bar{a}b\bar{c}) + (\bar{a}b\bar{c} + \bar{a}\bar{b}\bar{c})$$

$$= abc + \bar{a}\bar{b}c + ab\bar{c} + ab\bar{c} + \bar{a}b\bar{c} + \bar{a}\bar{b}\bar{c}$$

$$= ab(c + \bar{c}) + b\bar{c}(a + \bar{a}) + \bar{a}\bar{b}(c + \bar{c})$$

$$= ab + b\bar{c} + \bar{a}\bar{b}$$

(B) (3 points) Zoomba wants to implement Boolean function G(A, B, C) given by the truth table below. Find the normal form and a minimal sum-of-products expression for G(A, B, C).

A	В	C	G
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

1. Normal form for G =
$$a\bar{b}\bar{c} + a\bar{b}\bar{c} + a\bar{b}\bar{c} + a\bar{b}\bar{c} + ab\bar{c}$$

$$\bar{a}\bar{b}\bar{c} + \bar{a}b\bar{c} + a\bar{b}\bar{c} + a\bar{b}c + a\bar{b}c = \bar{c} + a\bar{b}c = \bar{c} + a\bar{b}$$

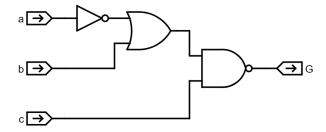
2. Minimal sum of products for
$$G = c + a\bar{b}$$

(C) (4 points) Draw the circuit that implements the function G using 3 or fewer gates. You may only use inverters and 2-input OR, NOR, AND, and NAND gates in your circuit. To receive any credit for this question, your circuit must implement the given truth table of function G, regardless of whether your minimal sum-of-product expression is correct or not.

 $\bar{c} + a\bar{b}$ alone can be constructed using 4 gates with using NOT / AND /OR, but if we use De Morgans rule, we can rewrite the expression as

$$\bar{c} + a\bar{b} = \overline{c \cdot (a\bar{b})} = \overline{c(b + \bar{a})}$$

Which can be constructed using 1 Not, 1 OR and 1NAND gate



(D) (3 points) Zoomba likes function G so much that they would like to construct all possible Boolean functions using G. Determine whether G is functionally complete. Explain your answer.

Yes, G is functionally complete

We can construct a NOT gate using G(0, 0, c), and we can construct an AND gate using G(a, G(0,0,b), 1).

Since {AND, NOT} is functionally complete, we can therefore conclude that a G gate alone is functionally complete.	

Problem 3. DrCMOS Will See You Now (19 points)

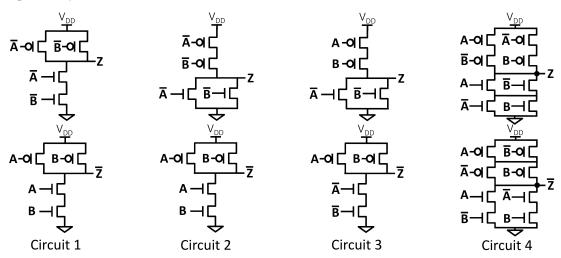
Ben Bitdiddle is disappointed with the limitation that CMOS logic is inverting: many common functions need multiple CMOS gates in series, which adds propagation delay.

To solve this problem, Ben proposes Dual-rail CMOS (or DrCMOS) logic. In a DrCMOS gate:

- Each input and output X is encoded using two wires (X, \overline{X}) that hold the value X and its complement, i.e., these wires can be (0, 1) or (1, 0). This is called dual-rail encoding.
- Each DrCMOS gate consists of two separate CMOS gates, which compute the output (Z, \overline{Z}) . One of the CMOS gates produces Z, and the other produces \overline{Z} . The CMOS gates can use any of the input wires as their inputs.

The figure to the right shows a DrCMOS buffer, which is built with two CMOS inverters in parallel. Whereas with the conventional (single-wire) signal encoding, a buffer needs two CMOS inverters in series, this dual-rail encoding allows using a single CMOS gate to get each output, which is faster! By using this dual-rail encoding throughout the circuit and using DrCMOS gates, Ben argues that we can implement faster circuits.

(A) (8 points) For each of the following circuits shown below, find whether the circuit is a valid DrCMOS gate. If so, specify its Boolean function **in sum-of-products form**. If not, briefly explain why.



Circuit 1: Z = F(A,B) = NONE or NONE and explain why it's not a DrCMOS gate.

Bottom CMOS gate computes AND, top computes OR; outputs are not complementary.

Circuit 2: Z = F(A,B) = A*B or NONE and explain why it's not a DrCMOS gate.

This i	saΓ)rCM	OS A	ND \mathfrak{g}	rate
11115 1	lo a L		$\mathbf{O} \mathbf{D} \mathbf{I} \mathbf{L}$	μ	zaw

Circuit 3: Z = F(A,B) = NONE or NONE and explain why it's not a DrCMOS gate.

The pullup and pulldown networks are not complementary → top and bottom components aren't even valid CMOS gates.

Circuit 4: $Z = F(A,B) = \underline{AB} + \overline{AB}\underline{B}$ or NONE and explain why it's not a DrCMOS gate.

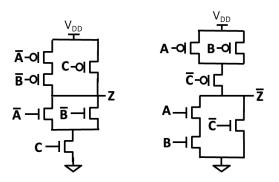
This is a DrCMOS XOR gate.

(B) (4 points) For each of the valid DrCMOS gates in part A, suppose that we swap the \mathbf{Z} and $\overline{\mathbf{Z}}$ signals (so that the CMOS gate at the top produces $\overline{\mathbf{Z}}$, and the one at the bottom produces \mathbf{Z}). What would be their new Boolean function? (Give each function in **sum-of-products form**.)

Circuit 2:
$$Z = \overline{AB} = \overline{A} + \overline{B}$$
 (a NAND gate)

Circuit 4:
$$Z = \overline{A}\overline{B} + \overline{A}B = AB + \overline{A}B$$
 (an XNOR gate)

(C) (4 points) A DrCMOS gate can be derived completely from just the pull-up or pull-down network of one of its two CMOS gates. To see this by example, the figure below shows the pull-down network of the CMOS gate that produces $\overline{\mathbf{Z}}$. Draw the rest of the DrCMOS gate. What is its Boolean expression?



Gate's Boolean expression (in sum-of-products form): $Z = F(A,B,C) = \underline{\qquad} AB + \overline{C}\underline{\qquad}$

You can compute the pull-up network for the \overline{Z} CMOS gate from the pulldown by using the normal CMOS rules to build a complementary pull-up (replace series with parallel subnetworks, and vice-versa).

For the \mathbf{Z} CMOS gate, starting from the $\overline{\mathbf{Z}}$ gate, make the networks complementary to those of the $\overline{\mathbf{Z}}$ gate, and use not(A) for each input A to each transistor.

(D) (3 points) Ben claims that any Boolean function can be implemented with a **single** DrCMOS gate. Is he right? If so, explain how to construct the gate from the Boolean expression. If not, give a counterexample, i.e., a Boolean expression that requires multiple DrCMOS gates.

Note: In your explanation, you can use the fact from part C that a single pull-up or pull-down network fully specifies the DrCMOS gate, and explain only how to implement one network from the Boolean expression.

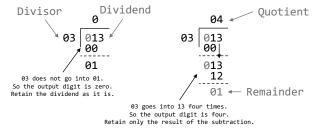
Yes, a single DrCMOS gate can implement any Boolean function.

To build the gate, we can start with the bottom CMOS gate, which produces $\overline{\mathbf{Z}}$. Put the Boolean expression for Z in sum-of-products form. If the expression has P product terms, the pull-down network is P parallel subnetworks, one for each product term. For each product term, if the term has E variables (negated or not), the corresponding subnetwork has E nFETs in series, each driven by one of the variables. For example, the product term $A\overline{B}$, would be implemented as 2 nFETs in series with one being driven by A and the other by \overline{B} . Once you have the pulldown for \overline{Z} , you can follow the procedure from part C to build the remaining 3 subnetworks.

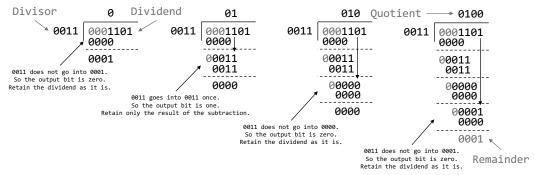
Problem 4: Back to 5th Grade (18 points)

Divya's favorite pastime is to quickly and efficiently divide integers. However, she is frustrated by the fact that CPUs rarely include combinational integer dividers. She wants to fix this problem once and for all by designing a combinational circuit that performs integer division. She remembers how binary operations often use algorithms learned in grade school, such as ripple carry addition for a + b and repeated shifted addition for $a \times b$, so long division cannot be that hard either, right? Let's help Divya implement integer division using grade school long division.

To divide two-digit decimal numbers, like $13 \div 03$, long division proceeds as follows:



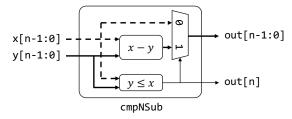
At each stage, we find out how many times the divisor goes into the current digits of the dividend, append that digit to the quotient and retain the remainder. Note that the highest digit of the remainder is always zero, so we ignore it. Indeed, $13 = 03 \times 04 + 01$, so the math checks out. We could use the same algorithm to perform this division in binary, i.e., $1101 \div 0011$:



(A) (4 points) Let's start off by thinking about a single comparison step. Implement a combinational circuit called cmpNSub that takes in two n-bit numbers, and returns the following:

$$cmpNSub(x,y) = \begin{cases} \{1'b1, x - y\}, & if \ y \le x \\ \{1'b0, x\}, & otherwise \end{cases}$$

The output of this circuit is an (n+1)-bit quantity, where the highest (n^{th}) bit represents the quotient bit, i.e., whether or not $y \le x$, and the lower n bits represent the current remainder, i.e., x - y or x, depending on the result of the comparison. A schematic for this circuit is as follows:



Complete the following function definition in Minispec.

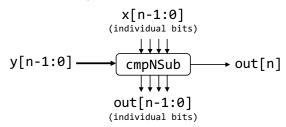
endfunction

(B) (2 points) What is the propagation delay of this function? Use $\Theta(.)$ notation. Assume that the subtractor is implemented with a Ripple-Carry Adder, and the comparator is implemented as a chain comparator, *i.e.*, both their delays scale as $\Theta(n)$. Provide a brief explanation.

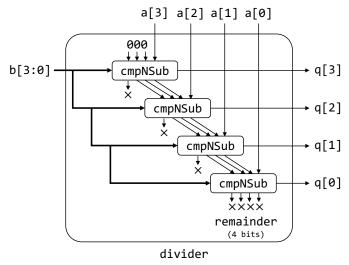
$$t_{pd}$$
 of an n-bit cmpNSub is $\Theta(\underline{\hspace{1cm}}\underline{\hspace{1cm}}\underline{\hspace{1cm}}\underline{\hspace{1cm}}n$.

Explanation: ___The critical path goes through the Ripple Carry Adder which is linear, or $\Theta(n)$.

(C) (10 points) Consider the following symbol for the cmpNSub circuit. The individual bits are shown on some signals for visual clarity of connections in the overall divider circuit.



We can connect n instances of cmpNSub in order to determine the final quotient and remainder. An example of the divider circuit is shown for n = 4 bits (\times indicates no connection):



Using the reference connections provided for n = 4 bits, implement a generalized divider function for two n-bit inputs and one n-bit output in Minispec that computes $a \div b$.

endfunction

(D) (2 points) Finally, using your work from parts (A), (B) and (C), calculate the propagation delay of this n-bit divider circuit. Provide a brief explanation.

t_{pd} of the n-bit divider is	$\Theta(\underline{\hspace{1cm}}^2\underline{\hspace{1cm}}).$

Explanation: ___ The critical path goes through n cmpNSubs in series, so $\Theta(n^2)$._____

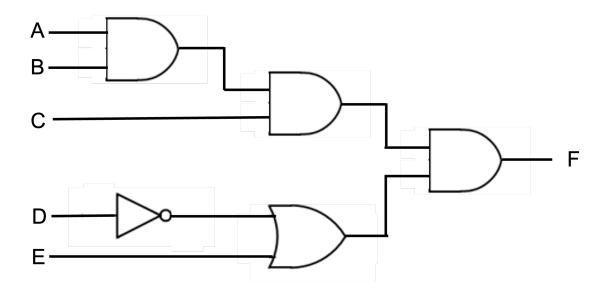
When you get your quiz back, try to synthesize the divider function for n = 32 and n = 64.

This will provide some insight into why combinational dividers are almost never included in processor ALUs, and are often entirely omitted out of lower-end processors!

Poor Divya will have to wait.

Problem 5. Combinational and Sequential Logic Timing (17 points)

(A) (4 points) Given the timing parameters of each logic gate in the table below, what are the propagation delay (t_{pd}) and contamination delay (t_{cd}) of the circuit below.

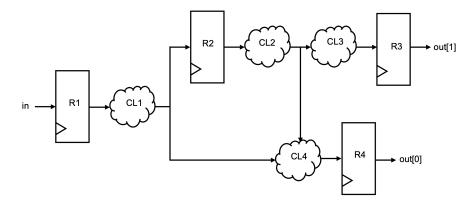


Gate	t_{pd}	t_{cd}
AND2	1.5ns	1ns
INV1	1ns	0.5ns
OR2	3ns	0.5ns

t_{pd} (ns): _____5.5____

t_{cd} (ns): _____1.5_____

For the rest of the problem, consider the sequential circuit below, as well as the timing specifications. Registers R1, R2, R3, and R4 are driven by a common clock with period t_{clk} = 15ns. CL1, CL2, CL3, and CL4 are combinational circuits.



	t_{pd}	t_{cd}	t _{setup}	t_{hold}
Register (R1, R2, R3, R4)	3ns	1ns	5ns	2ns
CL1	3ns	1ns		
CL2	2ns	1.5ns		
CL3	4ns	2ns		
CL4	5ns	1ns		

(B) (2 points) What are the t_{pd} and t_{cd} of this sequential circuit?

 t_{pd} (ns): _____3____

t_{cd} (ns): _____1____

(C) (4 points) Given the timing parameters in the table above, please indicate whether each constraint is satisfied by the circuit and explain your answer.

Setup time constraint (circle one): SA

SATISFIED

NOT SATISFIED

Explanation:

The path from R1 to R4 requires a minimum clock period of 16ns which is greater than the given clock period of 15ns.

R1 -> R4:
$$t_{PD,R1} + t_{PD,CL1} + t_{PD,CL4} + t_{SETUP,R4} = 3 + 3 + 5 + 5 = 16$$

Hold time constraint (circle one):

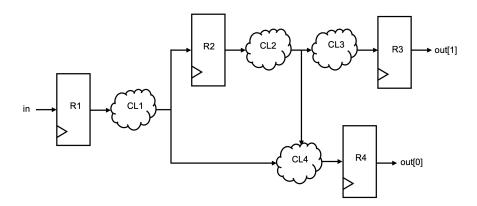
SATISFIED

NOT SATISFIED

Explanation: All register-to-register paths have sum of contamination delays adding up to be greater than or equal to the hold time of the final register.

- R1 -> R2: $t_{CD,R1} + t_{CD,CL1} = 1 + 1 = 2 = 2 = t_{HOLD,R2}$
- R1 -> R4: $t_{CD,CL1} + t_{CD,CL4} = 1 + 1 + 1 = 3 > 2 = t_{HOLD,R4}$
- R2 -> R3: $t_{CD,R2} + t_{CD,CL2} + t_{CD,CL3} = 1 + 1.5 + 2 = 4.5 > 2 = t_{HOLD,R3}$
- R2 -> R4: $t_{CD,R2} + t_{CD,CL2} + t_{CD,CL4} = 1 + 1.5 + 1 = 3.5 > 2 = t_{HOLD,R4}$

The sequential circuit and the original timing specifications have been copied here for your convenience.



	t_{pd}	t_{cd}	t_{setup}	t_{hold}
Registers	3ns	1ns	5ns	2ns
CL1	3ns	1ns		
CL2	2ns	1.5ns		
CL3	4ns	2ns		
CL4	5ns	1ns		

We now find that our supplier has the following alternative circuits for CL1, CL2, CL3, and CL4 available with the following specifications.

(D) (3 points) Our supplier replaces all registers with new registers with $t_{hold} = 2.5 \text{ns}$, all driven by a common clock with period $t_{clk} = 15 \text{ns}$. Your job is to make sure that the setup and hold time constraints are satisfied with these new registers. You may replace *only* one of the combinational circuits with its alternative specification. Please indicate which combinational circuit you are replacing. If the constraints are satisfied without any changes, please write **NONE**.

	t_{pd}	t_{cd}	t_{setup}	t_{hold}
Registers-New	3ns	1ns	5ns	2.5ns
CL1-New	2ns	1.5ns		
CL2-New	1ns	0.5ns		
CL3-New	3ns	2ns		
CL4-New	3ns	1ns		

Combinational circuit replaced (or NONE): ____CL1____

(E) (4 points) Show that your new circuit satisfies the setup and hold time constraints below. If no changes are needed, write **NO CHANGES REQUIRED** below.

Setup time constraint satisfied:

- R1-NEW -> R2-NEW: $t_{PD,R1-NEW} + t_{PD,CL1-NEW} + t_{SETUP,R2-NEW} = 3 + 2 + 5 = 10 < 15$
- R1-NEW -> R4-NEW: $t_{PD,R1-NEW} + t_{PD,CL1-NEW} + t_{PD,CL4} + t_{SETUP,R4-NEW} = 3 + 2 + 5 + 5 = 15 = 15$
- R2-NEW -> R3-NEW: $t_{PD,R2-NEW} + t_{PD,CL2} + t_{PD,CL3} + t_{SETUP,R3-NEW} = 3 + 2 + 4 + 5 = 14 < 15$
- R2-NEW -> R4-NEW: $t_{PD,R2-NEW} + t_{PD,CL2} + t_{PD,CL4} + t_{SETUP,R4-NEW} = 3 + 2 + 5 + 5 = 15 = 15$

Hold time constraint satisfied:

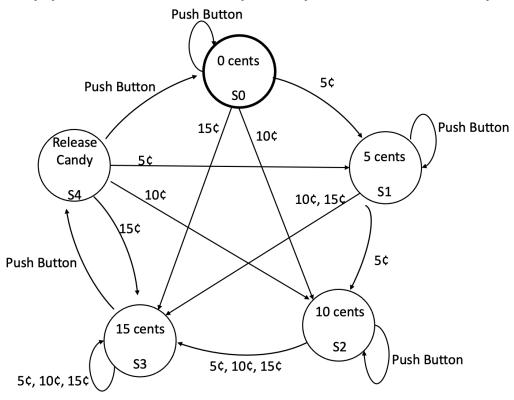
- R1-NEW -> R2-NEW: $t_{CD,R1-NEW} + t_{CD,CL1-NEW} = 1 + 1.5 = 2.5 = 2.5 = t_{HOLD,R2-NEW}$
- R1-NEW -> R4-NEW: $t_{CD,R1-NEW} + t_{CD,CL1-NEW} + t_{CD,CL4} = 1 + 1.5 + 1 = 3.5 > 2.5 = t_{HOLD,R4-NEW}$
- R2-NEW -> R3-NEW: $t_{CD,R2-NEW} + t_{CD,CL2} + t_{CD,CL3} = 1 + 1.5 + 2 = 4.5 > 2.5 = t_{HOLD,R3-NEW}$

• R2-NEW -> R4-NEW: $t_{\text{CD,R2-NEW}} + t_{\text{CD,CL2}} + t_{\text{CD,CL4}} = 1 + 1.5 + 1 = 3.5 > 2.5 = t_{\text{HOLD,R4-NEW}}$

Problem 6. Finite State Machines (15 points)

Sid the cookie monster is trying to get candy bars out of a vending machine. He has 5ϕ , 10ϕ , and 15ϕ coins, and one candy bar is worth 15ϕ . However, this machine is pretty rudimentary, and if more than 15ϕ are put in without exchanging it for candy, the vending machine will stay at 15ϕ and not record additional coins. If Sid tries to get a candy bar before putting in enough money, he won't get the candy but the amount recorded by the vending machine will stay the same.

To better understand how Sid can get candy, he draws the following finite state machine. Note that at every cycle of the FSM, there will always be an input of either a coin or a button press.



To represent this, Sid encodes the inputs into the machine using 2-bit values:

- Push Button: 2'b00
- 5¢: 2'b0110¢: 2'b10
- 15¢: 2'b11
- (A) (1 point) If the vending machine is in state S3 and Sid puts in any coin, what state does this FSM go to?

Next State:	S3
--------------------	-----------

(B) (1 point) What state must the vending machine be in for Sid to successfully get candy?

State:	S 4	
State:	34	

(1 point) How many flip flops does the vending machine FSM require to encode all possii states?				
Nu	umber of Flip Flops:	_3		
Before Sid starts using his hard-earned coins to get can machine will behave given a series of inputs.	dy, he wants to know how the	ne vending		
(D) (6 points) What is the resulting state (S0 through S each sequence of inputs provided? Assume the vercents , S0 , at the beginning of each sequence.	,			

i.	01 01 01 00	S4
ii.	10 11 01 00 10	S2
iii.	11 10 00 11 01 00 01 00 01 00	S2

(E) (6 points) Fill out the following truth table for Sid's finite state machine based on his state transition diagram. The output should be 1 whenever the FSM outputs a candy bar, and 0 otherwise.

State	Input	Next State	Output
S0	00 (Push Button)	S0	0
S0	01 (5¢)	S1	0
S0	10 (10¢)	S2	0
S0	11 (15¢)	S3	0
S1	00 (Push Button)	S1	0
S1	01 (5¢)	S2	0
S1	10 (10¢)	S3	0
S1	11 (15¢)	S3	0
S2	00 (Push Button)	S2	0
S2	01 (5¢)	S3	0
S2	10 (10¢)	S3	0
S2	11 (15¢)	S3	0
S3	00 (Push Button)	S4	0
S3	01 (5¢)	S3	0
S3	10 (10¢)	S3	0
S3	11 (15¢)	S3	0
S4	00 (Push Button)	S0	1
S4	01 (5¢)	S1	1
S4	10 (10¢)	S2	1
S4	11 (15¢)	S3	1

END OF QUIZ 1!